SPRING MANUFACTURERS' RESEARCH ASSOCIATION

The Design of Single Coil Spring Washers

Report No. 134

by

S.C. Gross

October 1962

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INTRODUCTION

Under the title "Contribution to the Stress and Strain Analysis of Helical Spring Washers", a short abstract from a report of the then Armament Design Establishment was published in the Coil Spring Journal 1954, No.36 pp. 32-24. The report was later published in full in the German journal Werkstatt und Betrieb 1956, pp 181-185.

Work on the revision of B.S.1802 "Steel Spring Washers" made it desirable to adapt the rather involved method in the original paper to practical design work. In fact, it proved possible to simplify it so much that the calculations for a spring washer design has become as simple as those for a helical spring.

In order to describe it, it will first be necessary to review the original one.

THE GEOMETRY OF THE WASHER

If a washer with mean diameter D and with rectangular cross-section of width W and thickness t is placed between the platens AA and BB of a load testing machine it takes the position as shown in Fig. 1 when a small load Q is applied, and only a small error will be incurred by assuming that Fig. 1 shows the washer in the unloaded condition;

The washer is in contact with each platen at two points C_1, C_2 and C'_1, C'_2, respectively, with distances p and q from the vertical axis NN' of the washer. The centre line PM_1M_2Q of the coil is a helix with pitch $PQ = h_o$ and diameter D, and its edges are also helices. Consequently, the projections of these helices on the plane of the elevation Fig. 1 are sine curves. The perpendicular dropped from point S on the outside upper edge of the coil onto its axis UU' which is also the axis of all the helices intersects with this axis at point O which may be chosen as the origin of system of co-ordinates (x,y). In this system, the curve corresponding to the outside upper edge of coil has the equation

$$x = \tfrac{1}{2}(D+ w)\ \cos\ (2\pi y/h_o). \tag{1}$$

Hence, the distance of C_1 from the y-axis

$$x_1 = \tfrac{1}{2}\ (D + w)\ \cos\ (2\pi y_1/h_o) \tag{1a}$$

where

where

$$2\pi y_1/h_o = \gamma \tag{2}$$

is the angular distance between S and C_1 (see the plan Fig. 1).

The distance of C_2 from the axis UU' equals half the width e of the slit, i.e. $x_2 = -e/2$, and it can be shown that the distance of this point from the x-axis is given by

$$y_2 = \frac{3}{4}h_o - \frac{t}{2} - \frac{e}{\pi D}\frac{h_o}{2} + \frac{t/2}{\cos\alpha}$$

$$= \frac{1}{2}\left[\left(\frac{3}{2} - \frac{e}{\pi D}\right)h_o + \left(\frac{1}{\cos\alpha} - 1\right)t\right]$$

where α is the helix angle the tangent of which is given by $\tan\alpha = h_o/\pi D$.

It remains to find y_1. The slope of the tangent to the curve (1) is given by

$$\frac{dy}{dx} = -\frac{h_o}{\pi(D+w)\sin(2\pi y/h_o)}$$

and its slope at C_1 by

$$\frac{dy}{dx}_{(x_1 y_1)} = -\frac{h_o}{\pi(D+w)\sin(2\pi y_1/h_o)} = \tan\beta \tag{3}$$

where β is the angle between the y-axis and the axis NN'.

On the other hand,

$$\tan\beta = \frac{y_2 - y_1}{x_2 - x_1},$$

and by equating these two terms for $\tan\beta$ and by substituting the terms for x_1, x_2, and y_2 we get

$$\frac{h_o}{\pi(D+w)\sin(2\pi y_1/h_o)} = \frac{\left(\frac{3}{2} - \frac{e}{\pi D}\right)h_o + \left(\frac{1}{\cos\alpha} - 1\right)t - 2y_1}{(D+w)\cos(2\pi y_1/h_o) + e} \tag{4}$$

As this equation is transcendental, y_1 can only be found by trial and error or graphically.

Once y_1 is known, x_1, γ and $\tan\beta$ can be calculated from (1(a), (2) and (3) and the following equations be derived

$$H_o = 2MN = 2MU\cos\beta$$

$$= \left[\left(1 - \frac{e}{\pi D}\right)h_o + \frac{t}{\cos\alpha}\right]\cos\beta - e\sin\beta, \tag{5}$$

$$p = \frac{x_1}{\cos\beta} - \frac{H_o}{2}\tan\beta, \tag{6}$$

$$q = \frac{H_o}{2}\tan\beta + \frac{e}{2\cos\beta}. \tag{7}$$

THE ELASTIC DEFORMATION

The washer load Q produces the reactions P_1 and P_2 at the points C_1, C_1' and C_2, C_2' respectively, which must satisfy the conditions of equilibrium.

$$P_1 + P_2 = Q \qquad \text{and} \qquad P_1 p = P_2 q.$$

$$\text{Hence} \quad P_1 = \frac{q}{p+q} Q \qquad \text{and} \qquad P_2 = \frac{p}{p+q}$$

These reactions can be resolved in components acting in the directions of the axes x and y, namely

$$V_1 = P_1 \cos\beta \qquad H_1 = P_1 \sin\beta \qquad V_2 = P_2\cos\beta \qquad H_2 = P_2 \sin\beta \tag{8}$$

Figs. 2 and 3 show the plan of the washer as seen against the positive direction of the y-axis. As always $e \ll \pi D$, the slit has been neglected, and the washer is considered a whole circle. Owing to the symmetry with respect to the diameter C_2M', only the semicircle C_2M_2M' need be dealt with which is divided in the two ranges of integration $\Phi_1 = 0$ —— $(\pi/2 - \gamma)$ and $\Phi_2 = (-\gamma)$ —— $\pi/2$ by point C_1' where V_1 is acting perpendicularly upwards.

Range 1 Torque T_1 and bending moments M_1 and M_1', acting at any point E_1.

$$T_1 = V_2 \times E_1K - V_1 \times E_1F_1 \qquad \text{(see Fig. 2)}$$

$$= V_2 \times \frac{D}{2}(1 + \cos\Phi_1) - V_1 \times \left[\frac{D}{2} - \frac{D+w}{2}\sin(\Phi_1 + \gamma)\right] \tag{9}$$

$$M_1 = V_2 \times C_2K - V_1 \times C_1' F_1 \qquad \text{(see Fig. 2)}$$

$$= V_2 \times \frac{D}{2}\sin\Phi_1 - V_1 \times \frac{D+w}{2}\cos(\Phi_1 + \gamma) \tag{10}$$

$$M_1' = H_2 \times E_1K - H_1 \times E_1F_1 \qquad \text{(see Fig. 3)}$$

$$= H_2 \times \frac{D}{2}(1-\cos\Phi_1) - H_1 \times \left(\frac{D}{2}\cos\Phi_1 - \frac{D+w}{2}\sin\gamma\right) \tag{11}$$

<u>Range 2</u> Torque T_2 and bending moments M_2 and M_2' acting at any point E_2

$$T_2 = V_2 \times E_2F_2 \qquad \text{(see Fig. 2)}$$

$$= V_2 \times \frac{D}{2}(1 - \sin\Phi_2) \tag{12}$$

$$M_2 = V_2 \times C_2F_2 \qquad \text{(see Fig. 2)}$$

$$= V_2 \times \frac{D}{2}\cos\Phi_2 \tag{13}$$

$$M_2' = H_2 \times C_2F_2 \qquad \text{(see Fig. 3)}$$

$$= H_2 \times \frac{D}{2}(1 - \sin\Phi_2) \tag{14}$$

The strain energy of the whole washer is given by

$$W = \frac{R}{GJ}\left[\int_0^{\pi/2-\gamma} T_1^2 d\Phi + \int_{-\gamma}^{\pi/2} T_2^2 d\Phi\right] + \frac{R}{EI_v}\left[\int_0^{\pi/2-\gamma} M_1^2 d\Phi + \int_{-\gamma}^{\pi/2} M_2^2 d\Phi\right] +$$

$$\frac{R}{EI_h}\left[\int_0^{\pi/2-\gamma} M_1'^2 d\Phi + \int_{-\gamma}^{\pi/2} M_2'^2 d\Phi\right] \tag{15}$$

It denotes J the resistance to twisting of a rectangular cross-section, and it is $I_v = wt^3/12$ and $I_h = w^3t/12$. The resistance to twisting J can be extracted from the formula for the spring rate S in B.S.1726 "Helical Compression Springs" if $\mu \times 10^6$ which includes the shear modulus $G = 11.5 \times 10^6$ lb/sq.in. of conventional spring steel is replaced by $\mu_o = \mu/11.5$. Then

$$J = \frac{\pi}{4}\mu_o w^2 t^2$$

The rather awkward solution of (15) is given in the original report and, for the purpose in hand, need not be repeated here. It may be written in the form

$$W = C\frac{Q^2D^3}{GJ}$$

where C is a dimensionless constant depending on the shape of the washer in the unloaded condition. By differentiating this equation we get

$$\frac{dW}{dQ} = 2\frac{CQD^3}{GJ} \tag{16}$$

and if we substitute $dW = Qd\delta$ where δ denotes the deflection,

$$\frac{d\delta}{dQ} = 2\frac{CD^3}{GJ} \quad \text{or} \quad \frac{dQ}{d\delta} = S = \frac{1}{2}\frac{GJ}{CD^3} = \text{const.} \tag{17}$$

The spring rate is constant, and the load deflection line is straight. This result had to be expected because C is a constant based on the shape of the washer in the unloaded condition. Actually, the washer changes its shape and the loading conditions considerably if it is loaded so that it may seem impermissible to rely on a straight load deflection line. However, load tests show that the load deflection line is straight and that the experimental spring rate is in good agreement with the result obtained from (17). Of course, there is a steep rise just before the washer is pressed flat, a phenomenon which also occurs with helical springs near closure of the coils.

If we assume that the load deflection line is straight until the washer is practically pressed flat so that its height equals or almost equals its thickness t, the preceding formulae may be drastically simplified. We may put

$$y_1 \longrightarrow 0; \quad \gamma \longrightarrow 0; \quad \beta \longrightarrow 0; \quad x_1 = p \longrightarrow \frac{D+w}{2}; \quad q \longrightarrow e/2$$

$$V_1 \longrightarrow \frac{e}{D+w+e}Q; \quad V_2 \longrightarrow \frac{D+w}{D+w+e}Q; \quad H_1 \longrightarrow 0; \quad H_2 \longrightarrow 0,$$

and, as $e << D+w$, e may also be neglected so that $V \longrightarrow 0$ and $V_2 \longrightarrow Q$. Then

$$T_1 = Q\frac{D}{2}(1 + \cos\Phi_1); \quad M_1 = Q\frac{D}{2}\sin\Phi_1; \quad M_1' = 0$$

$$T_2 = Q\frac{D}{2}(1 - \sin\Phi_2); \quad M_2 = Q\frac{D}{2}\cos\Phi_2; \quad M_2' = 0$$

Now we may raise the upper limit of Φ_1 from $\pi/2 - \gamma$ to π; then the equations for T_1 and M_1 cover the whole semicircle, and the equations for T_2 and M_2 become unnecessary. It remains

$$\left.\begin{aligned} T &= Q\frac{D}{2}(1 + \cos\Phi) \\ M &= Q\frac{D}{2}\sin\Phi \end{aligned}\right\} \tag{18}$$

The distribution of T and M over the circumference of the washer is shown in Fig. 4. The torque rises from zero at the slit (point C_1) to its maximum value Q x D at the diametrically opposite point M'. The bending moment is zero at both points and has its maximum value Q x D/2 at M_2. The maximum torque is twice as large as the torque in a helical spring with the same coil dimensions under the same load Q.

The Simplified Method of Calculation

If the simplifications mentioned are applied to (15), this equation becomes

$$W = Q^2R^3\left[\int_0^{\pi}\frac{(1+\cos\Phi)^2}{GJ}\,d\Phi + \int_0^{\pi/2}\frac{\sin^2\Phi}{EI_v}\cdot d\Phi\right]$$

$$= Q^2R^3 \left[\frac{3}{2}\frac{\pi}{GJ} + \frac{1}{2}\frac{\pi}{EI_v}\right]$$

$$= \frac{\pi}{16}\frac{Q^2D^3}{G}\left[\frac{3}{J} + \frac{G/E}{I}\right]$$

or, after substituting $J = \frac{\pi}{4}\mu_o w^2t^2 = \frac{\pi}{4}\frac{\mu_o}{m}w^3t$ and $I_v = \frac{wt^3}{12} = \frac{w^3t}{12m^2}$ where

$m = w/t$,

$$W = \frac{3}{4}\pi\frac{Q^2D^3}{G\ w^3}\ \frac{m}{t}\left[\frac{1}{\pi\mu_o} + mG/E\right]$$

and, if the coil index $c = D/w$ is introduced, finally

$$W = \frac{3}{4}\pi\frac{Q^2mc^3}{G\ t}\left[\frac{1}{\pi\mu_o} + mG/E\right]$$

$$= \frac{3}{4}\pi\frac{Q^2m^3c^3}{G\quad w}\left[\frac{1}{\pi m\mu_o} + G/E\right]$$

On the other hand, the strain energy is also given by $\frac{1}{2}Q\delta$. Therefore,

$$\frac{3}{4}\pi\frac{Q^2m^3c^3}{G\quad w}\left[\frac{1}{\pi m\mu_o} + G/E\right] = \frac{1}{2}Q\delta$$

and hence

the spring rate $$S = \frac{Q}{\delta} = \frac{2\,w\,G}{3\pi c^3m^3\left[\frac{1}{\pi m\mu_o} + G/E\right]} \tag{19}$$

and the load $$Q = \frac{2wG\delta}{3\pi c^3m^3\left[\frac{1}{\pi m\mu_o} + G/E\right]} . \tag{20}$$

For conventional spring steel with $G = 11.5 \times 10^6$ lb/sq.in. and $G/E = 11.5/30 = 0.3833$, these formulae turn in

$$S = \frac{6.37 \times 10^6}{m^3\left[1 + \frac{0.830}{m\mu_o}\right]}\ \frac{w}{c^3} \tag{19a}$$

$$Q = \frac{6.37 \times 10^6}{m^3\left[1 + \frac{0.830}{m\mu_o}\right]}\ \frac{w\delta}{c^3} = S\delta \tag{20a}$$

or, if the abbreviation

$$\frac{6.37}{m^3\left[1 + \frac{0.830}{m\mu_o}\right]} = A \tag{21}$$

is introduced, in

$$S = A\frac{w}{c^3} \times 10^6 \tag{19b}$$

$$Q = A\frac{w\delta}{c^3} \times 10^6$$

$$= S\delta \tag{20b}$$

The load factor A which is a function of m [x] is tabledin Table 1 and plotted against m in the graph Fig. 5.

Table 1 Load Factor A

m	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.0	2.2	2.4
A	1.130	0.913	0.751	0.621	0.517	0.436	0.372	0.274	0.207	0.160	0.126

Table 2 Deflection Factor B

m(c+1)	5	5.5	6	7	8	9	10	11	12
n					B				
1.0	0.919	0.932	0.943	0.958	0.967	0.974	0.979	0.983	0.985
1.1	0.999	1.014	1.028	1.046	1.059	1.067	1.073	1.078	1.081
1.2	1.075	1.095	1.110	1.134	1.149	1.159	1.167	1.173	1.777

The terms in (5) which contain e may be omitted as before, and cos α differs so little from unity that (5) may be written

$$H_o = (h_o + t)\cos\beta$$

The pitch h_o can be related to t by putting $h_o = nt$ where $n \gtrless 1$. Then

$$H_o = (n+1)t\cos\beta \tag{22}$$

As

$$\cos\beta = \frac{1}{\sqrt{1 + \tan^2\beta}}$$

we get by substituting $\tan\beta$ from (3)

$$\cos\beta = \frac{1}{\sqrt{1+\left[\dfrac{h_o}{\pi(D+w)\sin(2\pi y_1/h_o)}\right]^2}} = \frac{1}{\sqrt{1+n^2\left[\dfrac{t}{\pi w(c+1)\sin(2\pi y_1/h_o)}\right]^2}}$$

$$= \frac{1}{\sqrt{1+n^2\dfrac{1}{[\pi\, m(c+1)]^2\sin^2(2\pi y_1/h_o)}}}$$

---

[x] In B.S.1726:1951, the coefficient μ is given as a function of the side ratio m and of the relative curvature c of the coil called the spring index. However, it has been found that the values of the spring rate as calculated and as measured are in better agreement if the influence of the curvature on μ is neglected by exclusively using the values of μ for $c = \infty$ (zero curvature) whatever the actual value of c may be. Therefore, in the draft of the revised version of B.S.1726 it is suggested that the new coefficient $\mu/11.5 = \mu_o$ be based only the μ-values for $c = \infty$.

or, as $1/\sin^2(2\pi y_1/h_o) = 1 + \cot^2(2\pi y_1/h_o)$,

$$\cos\beta = \frac{1}{\sqrt{1+ n^2\dfrac{1+\cot^2(2\pi y_1/h_o)}{[\pi m(c+1)]^2}}}$$

The term $\cot^2(2\pi y_1/h_o)$ is obtainable from equation (4) which by neglecting terms with e and by putting $1/\cos\alpha \doteq 1$, becomes

$$\frac{h_o}{\pi\sin(2\pi y_1/h_o)} = \frac{{}^3/_2\, h_o - 2y_1}{\cos(2\pi y_1/h_o)}$$

or

$$2\pi y_1/h_o+\cot(2\pi y_1/h_o) = \frac{3}{2}\pi$$

It can be found by trial and error that this equation is satisfied by $2\pi y_1/h_o = 12^\circ 32'35''$ and $\cot(2\pi y_1/h_o) = 4.493$. Therefore, as $\cot^2(2\pi y_1/h_o) \simeq 20.2$,

$$\cos\beta = \frac{1}{\sqrt{1 + \dfrac{21.2\ n^2}{[\pi m(c+1)]^2}}} = \frac{1}{\sqrt{1 + \dfrac{2.15\ n^2}{[m(c+1)]^2}}} \tag{23}$$

The maximum deflection is given by

$$\delta = H_o - t$$

or, if (22) and (23) are used, by

$$\delta = \left[\frac{n + 1}{\sqrt{1+2.15\ n^2/[m(c+1)]^2}} - 1\right] t$$

If we put

$$\frac{n + 1}{\sqrt{1 + 2.15\ n^2/ [m(c+1)]^2}} - 1 = B$$

we get

$$\delta = Bt \tag{24}$$

The deflection factor B for n = 1, 1.1 and 1.2 is tabled in Table 2 and plotted against m(c+1) in the graph Fig. 6.

If $\delta = Bt$ is substituted in (20b), this equation reads

$$Q = AB\,\frac{wt}{c^3} \times 10^6$$

$$= BSt \tag{20c}$$

For the maximum shear stress, B.S.1726 gives the formula

$$q = \lambda\frac{P}{bh}$$

or, by using the notation of this report,

$$q = \lambda\frac{Q}{wt}$$

where $\lambda = (m+1)(c+1)\lambda_1$ is given in a design chart and λ_1 in a table.

As the load is acting in the axis of a helical spring, the stress is produced by the torque Q x R whereas the washer is subject to the maximum torque Q x D = 2QR. Therefore the maximum stress in a washer is twice as large as in a helical spring, namely

$$q = 2\lambda\frac{Q}{wt}$$

$$\text{or}\quad q = 2\lambda\frac{AB}{c^3} \times 10^6 \qquad (25)$$

Numerical Examples

1. The 5/8" spring steel washer on Table 4 of B.S.1802:1951 has the following nominal dimensions:

Outside dia. D_o = 1.072 in. Width w = 0.192 in.
Mean thickness t = 0.128 in.

Hence $D = D_o - w = 1.072 - 0.192 = 0.880$ in., $m = \frac{w}{t} = \frac{0.192}{0.128} = 1.5$,

$c = \frac{D}{w} = \frac{0.880}{0.192} = 4.58$, $c^3 = 96.1$, $m(c+1) = 1.5 \times 5.58 = 8.37$

For m = 1.5, from Fig. 5 or Table 1: A = 0.436
For m(c+1) = 8.37 and for n = 1, 1.1, and 1.2 from Fig. 6: B = 0.985, 1.062 and 1.153, respectively.
For m = 1.5 and c = 4.58 from Table II in B.S.1802: λ_1 = 1.22.
Therefore, $\lambda = (m+1)(c+1)\lambda_1 = 2.5 \times 5.58 \times 1.22 = 17.$
From (19b): $S = \frac{Aw}{c^3} \times 10^6 = \frac{0.436 \times 0.192}{96.1} 10^6 = 872$ lb/in.

From (24): $\delta = Bt = 0.128B$
From (20c): $Q = BSt = 872 \times 0.128B = 111.5B$ lb
From (25): $q = 2\lambda\frac{AB}{c^3} 10^6 = 2 \times 17 \frac{0.436}{96.1} 10^6 B = 154000B$

n	B -	δ in.	Q lb	q lb/sq.in.
1.0	0.985	0.126	110	151500
1.1	1.062	0.136	118.5	163500
1.2	1.153	0.148	128.5	177500

2. For comparison, maximum load and stress of the 5/8" heavy range washer in the American standard ASA B.27.1-1958 may be calculated. The nominal dimensions are:

Inside dia. D_i = 0.648 in., w = 0.210 in., t = 0.189 in.
Then $D = D_i + w = 0.858$ in., m = 1.1, c = 4.083, c^3 = 68.05,
m(c+1) = 5.64, A = 0.900, B = 0.945, 1.0185, 1.0995, λ_1 = 1.28 λ = 13.74

$S = \frac{0.900 \times 0.210}{68.05} = 2780$ lb/in., $\delta = 0.189B$, $Q = 2780 \times 0.189B$,

$q = 2 \times 13.74 \frac{0.900}{68.05} 10^6 B = 363000B.$

n	B -	δ in.	Q lb	q lb/sq.in.
1.0	0.945	0.1785	497	343000
1.1	1.0185	0.1925	536	370000
1.2	1.0995	0.208	578	399000

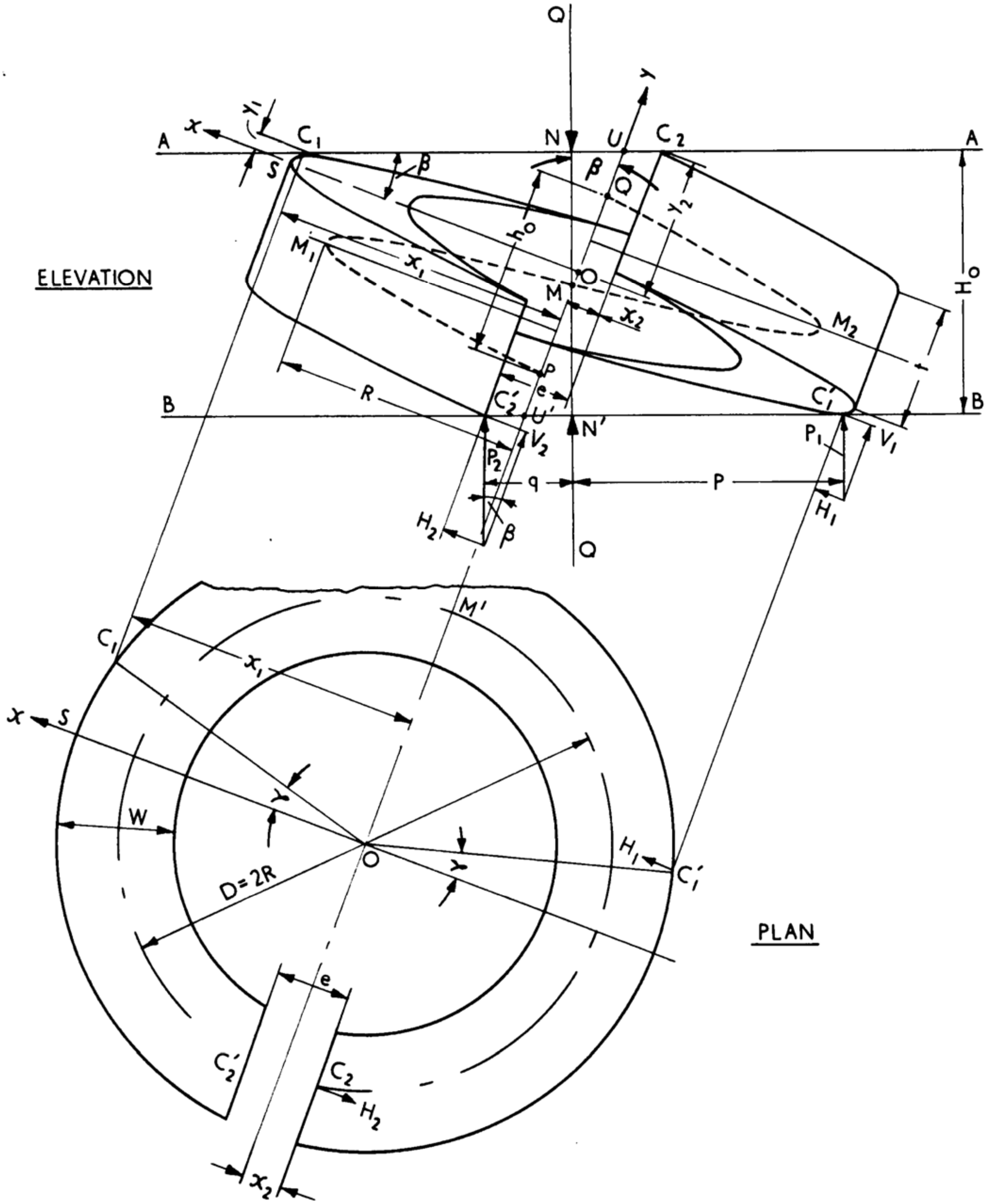
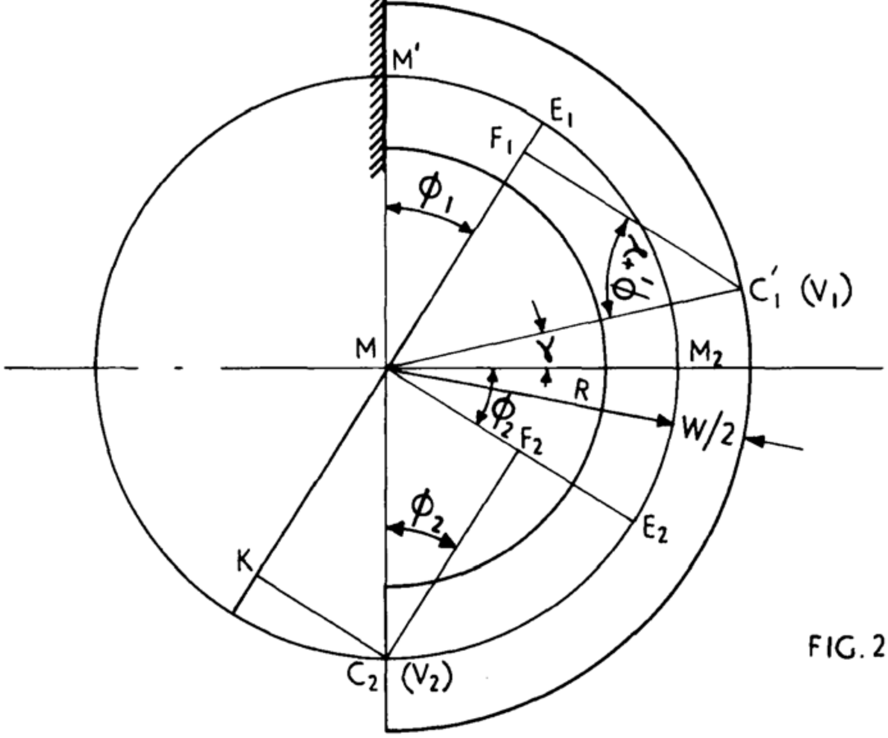
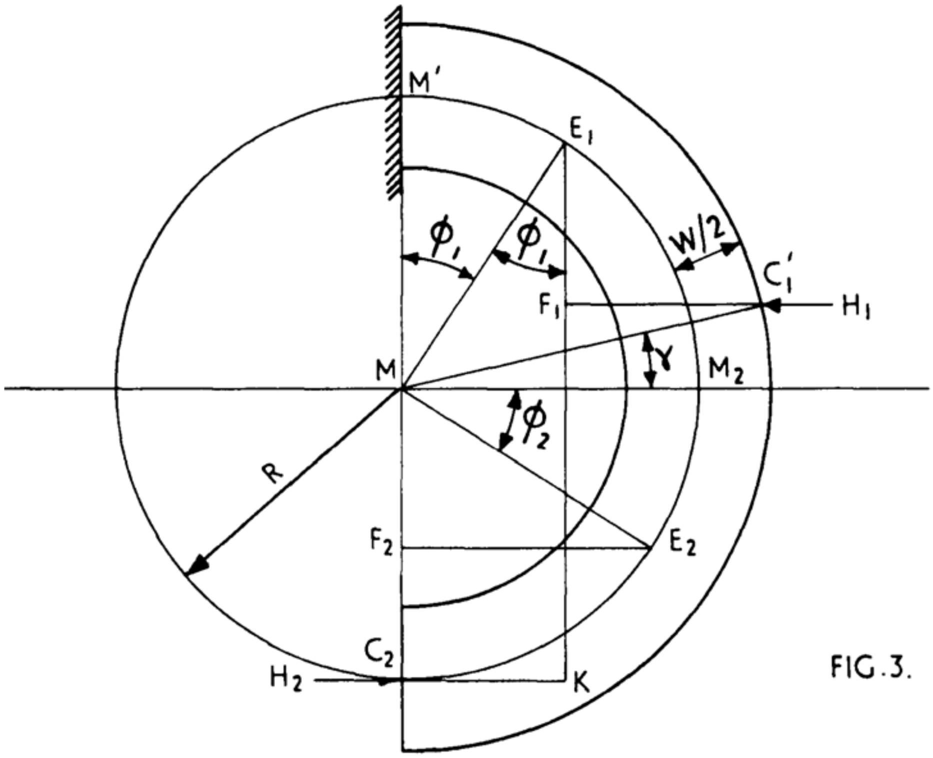

ELEVATION
PLAN
Q
A
N
U
C_2
C_1
S
β
h_o
y_2
M_1
x_1
O
M
x_2
M_2
H_o
R
P
e
C_2'
U'
C_1'
B
N'
V_2
P_2
P_1
V_1
q
H_1
H_2
M'
W
D=2R
γ

FIG. 1.



M'
E1
F1
φ1
φ1+γ
C'1 (V1)
M
γ
M2
R
φ2
F2
W/2
E2
φ2
K
C2 (V2)

FIG. 2.



M'
E1
φ1
φ1
W/2
C'1
F1
H1
M
γ
M2
φ2
R
F2
E2
C2
H2
K

FIG. 3.

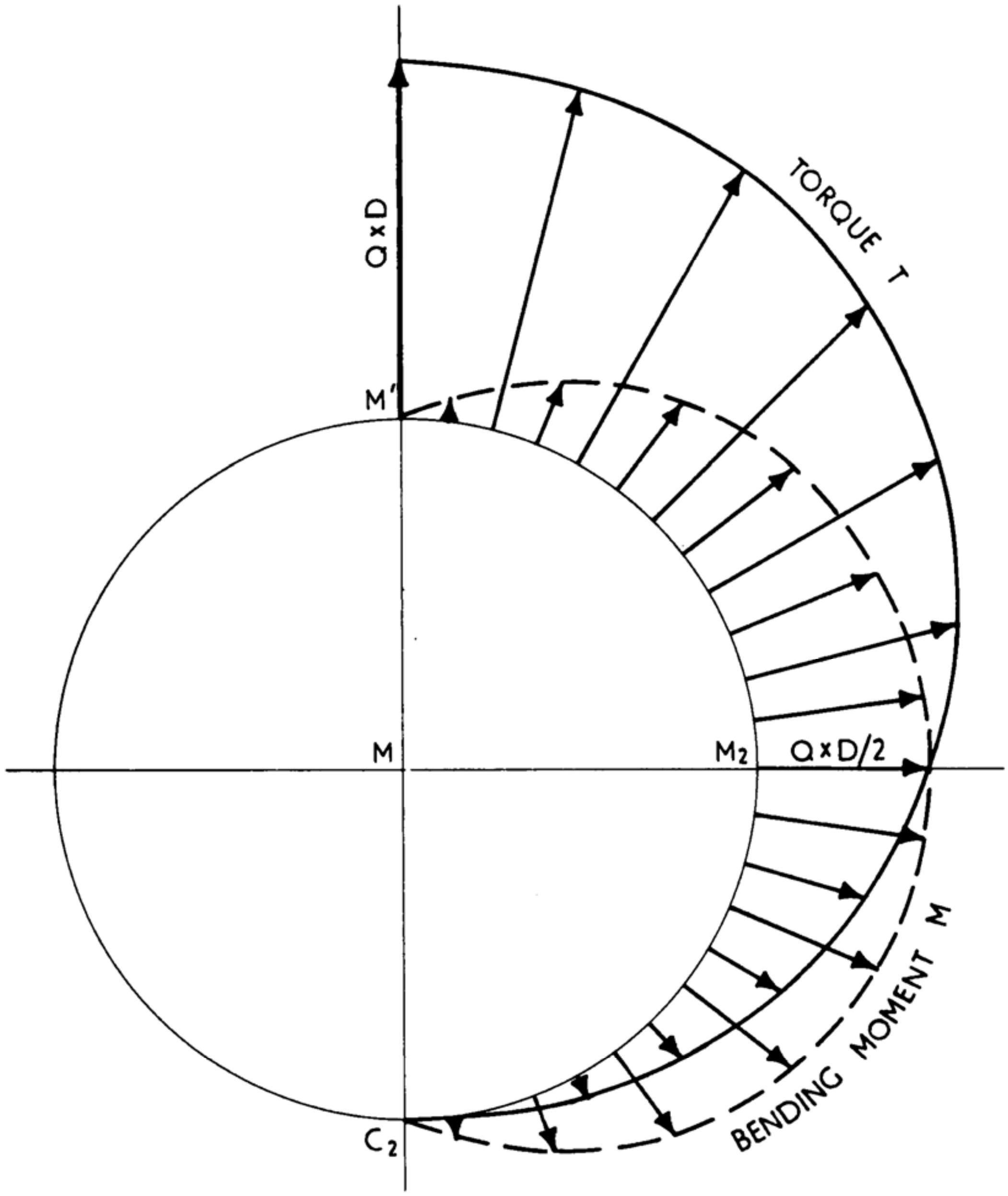


FIG. 4.

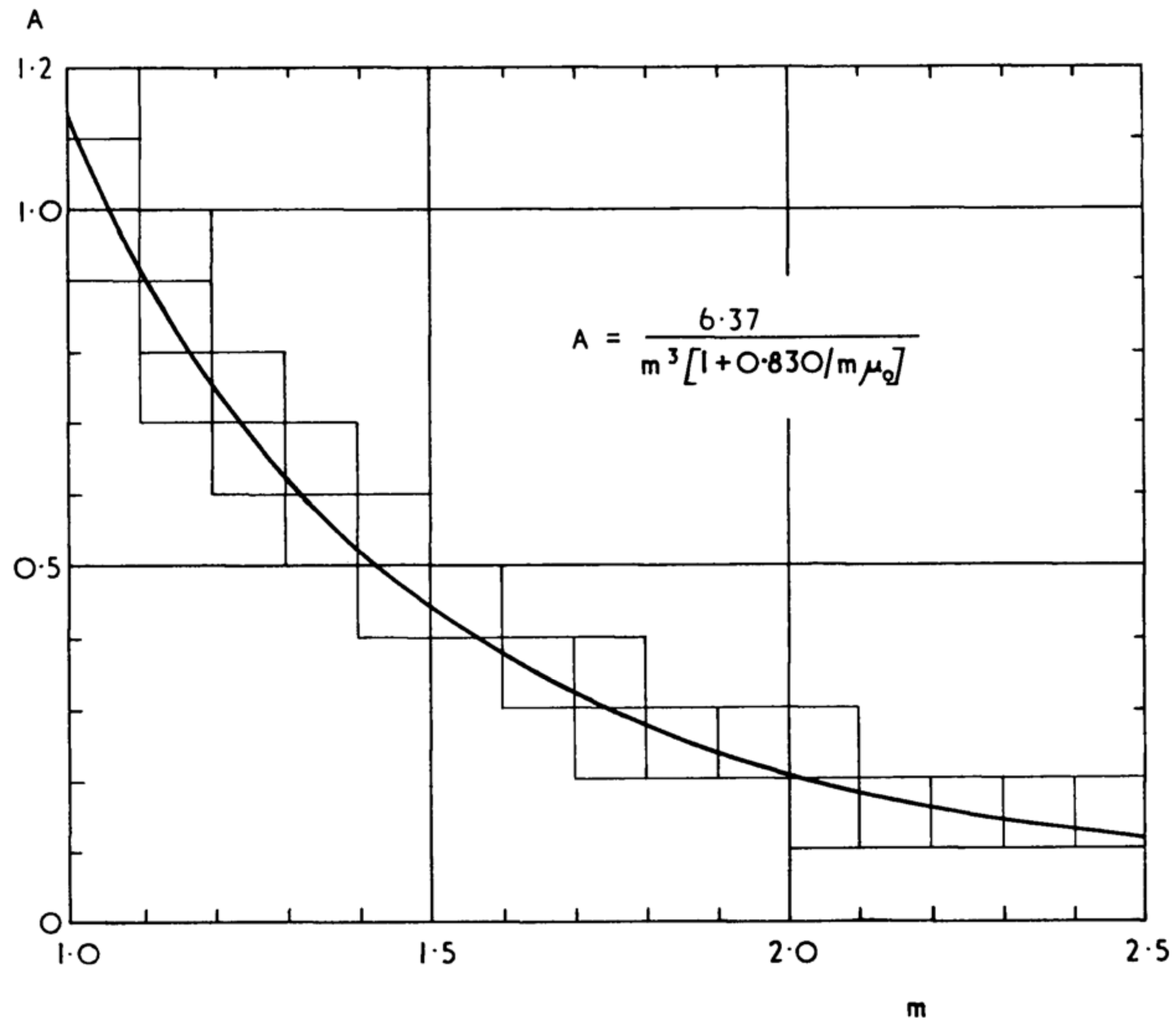


FIG. 5.

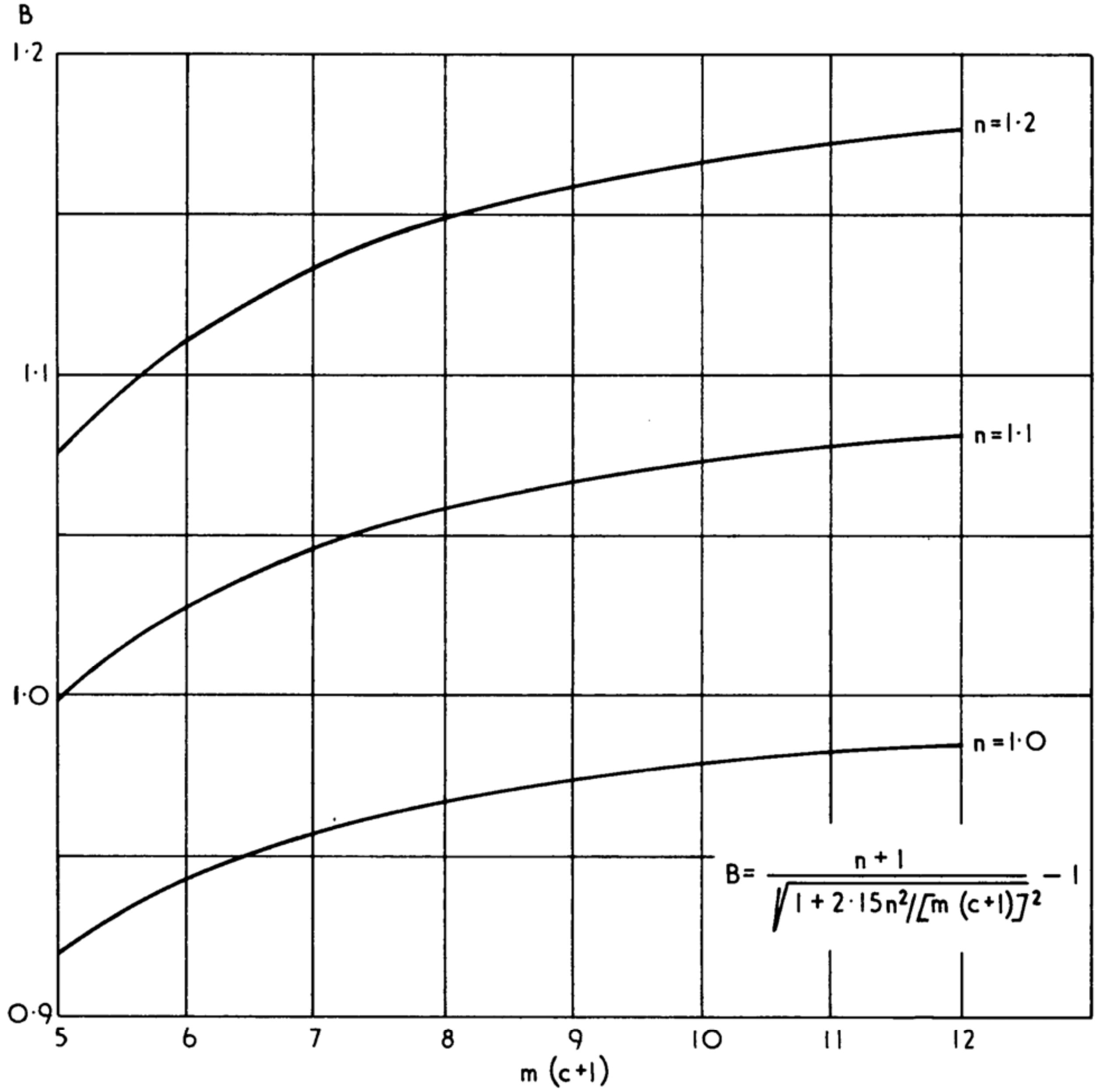

B
1·2
1·1
1·0
0·9
5
6
7
8
9
10
11
12
m (c+1)
n=1·2
n=1·1
n=1·0
$B = \frac{n+1}{\sqrt{1 + 2{\cdot}15 n^2/[m(c+1)]^2}} - 1$

FIG. 6.